SECTION 16.4: TRIPLE INTEGRALS

RECALL: Given a planar region R and continuous function f:

$$\iint_R f(x,y)\,dA$$

meant to 'chop up' and 'add up' in two directions. Here, $dA = dy\,dx$, $dA = dx\,dy$, or $dA = r\,dr\,d\theta$.

TRIPLE INTEGRAL: Given a solid Q and continuous function F:

$$\iiint_Q F(x,y,z)\,dV$$

means to 'chop up' and 'add up' in **three** directions.

Here, $dV = dz\,dy\,dx$, $dV = dz\,dx\,dy$, $dV = dy\,dz\,dx$,$dV = dy\,dx\,dz$, $dV = dx\,dz\,dy$, $dV = dx\,dy\,dz$.

EXAMPLE 1: Consider: $\displaystyle\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} (2z - x)\,dz\,dy\,dx$

1. Sketch or otherwise describe the solid Q over which this integral is being evaluated.

2. Evaluate this integral.

Ans: $\displaystyle\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} (2z - x)\,dz\,dy\,dx = 8$

VOLUME USING A TRIPLE INTEGRAL:

RECALL: For a planar region R, we have that: Area of $R = \iint_R 1\, dA$.

Hence, for a solid Q in space, we have: Volume of $Q = \iiint_Q 1\, dV$

EXAMPLE 2: Set-up a triple iterated integral with the prescribed integration order which computes the volume of the tetrahedron bounded by $x + 2y + 3z = 6$ in the first octant.

1. $dV = dz\, dy\, dx$

Ans: $\int_0^6 \int_0^{-\frac{1}{2}x+3} \int_0^{-\frac{1}{3}x-\frac{2}{3}y+2} 1\, dz\, dy\, dx = 6\,\text{units}^3$

2. $dV = dx\, dy\, dz$

Ans: $\int_0^2 \int_0^{-\frac{3}{2}z+3} \int_0^{-2y-3z+6} 1\, dx\, dy\, dz = 6\,\text{units}^3$

3. $dV = dy\, dx\, dz$

Ans: $\int_0^2 \int_0^{-3z+6} \int_0^{-\frac{1}{2}x-\frac{3}{2}z+3} 1\, dy\, dx\, dz = 6\,\text{units}^3$

EXAMPLE 3: Rewrite the integral $\int_0^4 \int_0^1 \int_{2-\sqrt{4-z}}^{2+\sqrt{4-z}} xy\, dx\, dy\, dz$ with order $dy\, dz\, dx$ and evaluate.

Ans: $\int_0^4 \int_0^1 \int_{2-\sqrt{4-z}}^{2+\sqrt{4-z}} xy\, dx\, dy\, dz = \int_0^4 \int_0^{-x^2+4x} \int_0^1 xy\, dy\, dz\, dx = \frac{32}{3}$

AVERAGE VALUE

RECALL: The average value of f over a region R is:

$$\overline{f} = \frac{1}{\text{Area of } R} \iint_R f(x,y)\, dA$$

DEFINITION: The average value of F over a solid Q is:

$$\overline{F} = \frac{1}{\text{Volume of } Q} \iiint_Q F(x,y,z)\, dV$$

EXAMPLE 4: Suppose the temperature at a point in space is given by $T(x,y,z) = xyz$.

Find the average temperature of the block described by: $Q = \{(x,y,z) : 0 \leq x \leq 1,\ 0 \leq y \leq 2,\ 0 \leq z \leq 3\}$

Ans: $\overline{T} = \dfrac{1}{6}\displaystyle\int_0^1 \int_0^2 \int_0^3 xyz\, dz\, dy\, dx = \dfrac{3}{4}$

HOMEWORK: Section 16.4: 7 - 63 every other odd.